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15.1 Double Integral over Rectangles

Recall Area problem $\Longrightarrow$ Definite integral

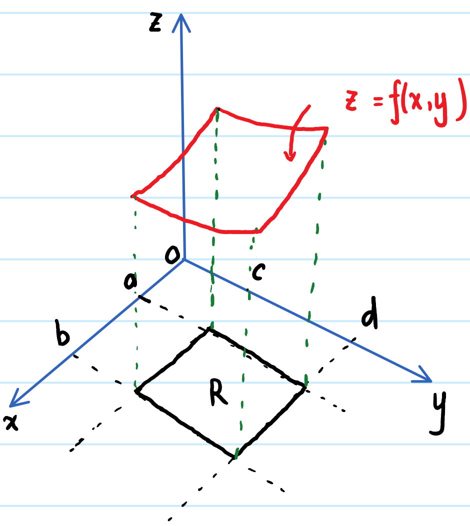
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

IDEA : Volume problem $\Longrightarrow$ Double Integral.

Volumes and Double Integrals

A (closed) rectangle $R = [a,b] \times [c,d]$, is the set of all points $(x,y) \in \mathbb{R}^2$ such that $a \le x \le b$ and $c \le y \le d$.

Let $f(x,y) \ge 0$ a continuous function defined on R. Let S denote the solid under the graph $z = f(x,y)$ over the region R.



Goal : Find volume of $S = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R,\ 0 \le z \le f(x,y) \}$

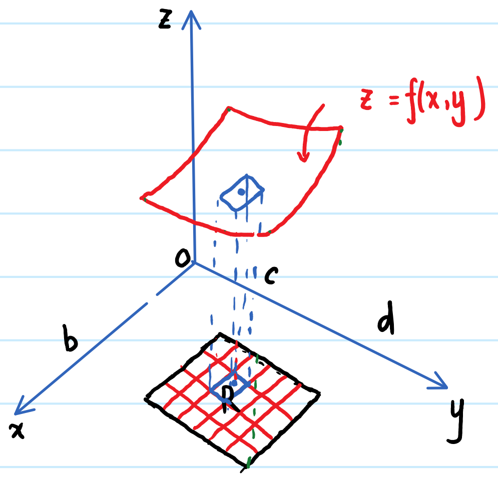
Riemann Sum Divide $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{m}$ and $[c,d]$ into n subintervals $[y_{J-1}, y_J]$

of equal width $\Delta y = \frac{d-c}{n}$.

So we divide R into a union of smaller rectangles

$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, each with area $\Delta A = \Delta x . \Delta y$.

Pick an arbitrary point (sample point) (x^*_{ij}, y^*_{ij}) in each R_{ij}, and then we can approximate the part of S that lies above R_{ij} by a thin rectangular box with base R_{ij} and height $f(x^*_{ij}, y^*_{ij})$



The repeating this procedure for all rectangles, we get an approximation for the volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A = S_{mn}$$

This approximation gets better and better as m & n become larger we expect that

$$V = \lim_{m,n \to \infty} S_{mn} = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A$$

DEF The double integral of f over a rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \quad \text{provided the limit exists.}$$

Thm Let f be a bounded function defined on a rectangle R.
Assume the set of all points where f is not continuous consists of a finite number of continuous curves and/or points.

Then f is integrable over R.

Property of Double Integrals

Let f, g be integrable functions on R and let c be a constant. Then

a) The function $f+g$, cf are both integrable and

$$\iint_R f+g\, dA = \iint_R f\, dA + \iint_R g\, dA$$

$$\iint_R cf\, dA = c\iint_R f\, dA$$

b) If $f(x,y) \leq g(x,y)$ for all (x,y) in R, then

$$\iint_R f\, dA \leq \iint_R g\, dA.$$

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Iterated Integrals

For a single integrals , FTC provides an easier way of computing integrals.

Double Integrals : Express a double integral as an iterated integral, which can be evaluated by calculating two single integrals.

Let $f(x,y)$ be integrable on $R = [a,b] \times [c,d]$.

$\int_c^d f(x,y)\,dy \equiv$ x is held fixed and $f(x,y)$ is integrated wrt y from $y = c$ to $y = d$.

partial integration wrt y.

$A(x) = \int_c^d f(x,y)\,dy$ is a number that depends on x, so defines a function.

Now we can integrate a function A wrt x from $x = a$ to $x = b$, obtaining

$$\int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx}_{\text{iterated integral}}$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

first integrate wrt y from $y = c$ to $y = d$ (while holding x fixed) and then integrate resulting function of x wrt from $x = a$ to $x = b$

Evaluate

1) $$\int_1^2\int_0^2 (x-3y^2)\,dx\,dy = \int_1^2\left[\int_0^2 x-3y^2\,dx\right]dy = \int_1^2\left[\frac{x^2}{2}-3xy^2\right]_0^2 dy$$

$$= \int_1^2 (2-6y^2)\,dy = 2y-2y^3\Big]_1^2 = -12.$$

2) $$\int_0^2\int_1^2 (x-3y^2)\,dy\,dx = \int_0^2\left[xy-y^3\right]_1^2 dx = \int_0^2 x-7\,dx = \frac{x^2}{2}-7x\Big]_0^2 = -12$$

Remark Notice that we got the same answer whether we integrated wrt y or x first. In general, it turns out that the two iterated integrals are always equal. (i.e. order of integration does not matter).

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Fubini's Theorem

If f is continuous on the rectangle $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

Ex Evaluate $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA$, $R=[0,1]\times[-3,3]$

$$\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{x\,y^3/3}{x^2+1}\right]_{-3}^3 dx = \int_0^1 \frac{18x}{x^2+1}\,dx$$

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$$= \frac{18}{2} \ln|x^2+1| \Big]_0^1 = 9 \ln|2|.$$

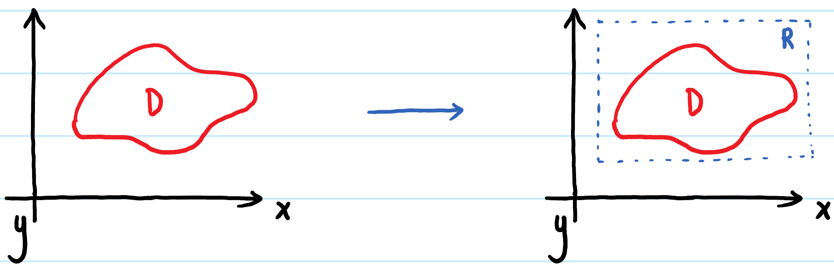
<u>Remark</u> : $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx \int_{-3}^{3} y^2\,dy = \int_0^1\int_{-3}^{3} \frac{xy^2}{x^2+1}\,dy\,dx$

In general, if $f(x), g(y)$ are both continuous function on $R = [a,b]\times[c,d]$, then

$$\iint_R f(x)g(y)\,dA = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right).$$

<u>15.3</u> Double Integrals over General Regions :

Idea is we want to be able to integrate a function f not just over rectangles but also over regions D over more general shape.



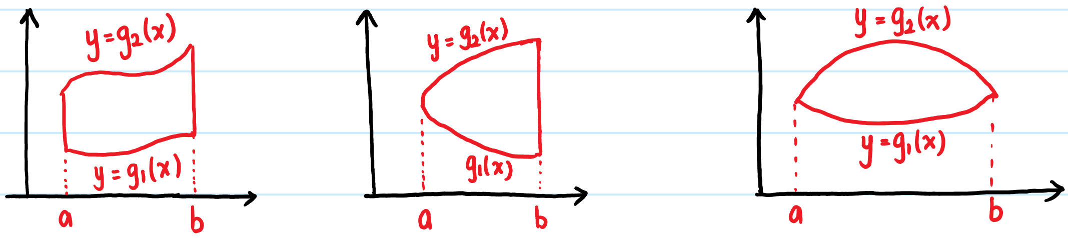
<u>Def</u> Let $f(x,y)$ be a continuous function defined on a closed and bounded region $D\subset\mathbb{R}^2$. Then $\iint_D f\,dA$ over D is given by

$$\iint_D f\,dA = \iint_R F\,dA \quad \text{where,}$$

R is a large rectangle containing D and $F(x,y) = \begin{cases} f(x,y) & , (x,y)\in D \\ 0 & , (x,y)\in R-D \end{cases}$

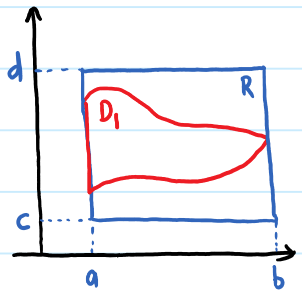
Regions of Type 1 and 2

Type 1 : $D_1 = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$



In order to evaluate $\iint_{D_1} f(x,y)\,dA$, when D is a region of type I,

choose $R = [a,b] \times [c,d]$ that contains D_1



$$\iint_{D_1} f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

Note that $F(x,y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ since (x,y) lies outside D.

Thus $\displaystyle\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$ since $F(x,y) = f(x,y)$ when $g_1(x) \leq y \leq g_2(x)$.

Therefore, if f is continuous on a Type I region D_1 such that

$D_1 = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_{D_1} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

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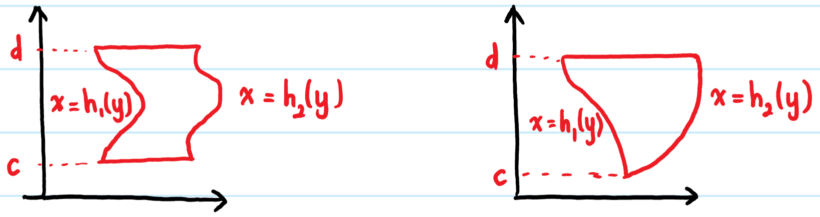
Type 2 : $D_2 = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$

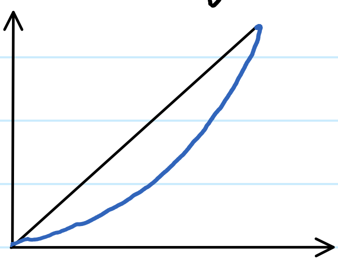


Similar reasoning as above can be used to show the following :
Therefore, if f is continuous on a Type II region D such that

$D_2 = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example Evaluate the integral $\iint_D xy \ dA$, where D is bounded by $y = x$, $y = x^3$, $x \geq 0$.



Type I Region

$D = \{(x,y) \mid 0 \leq x \leq 1,\ x^3 \leq y \leq x\}$

$$\iint_D xy \ dA = \int_0^1 \int_{x^3}^{x} xy \ dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_{x^3}^{x} dx = \int_0^1 \frac{x^3}{2} - \frac{x^7}{2}\,dx$$

$$= \left[\frac{x^4}{8} - \frac{x^8}{16}\right]_0^1 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}.$$

Properties of Double Integrals

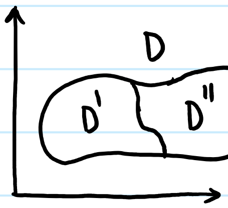
1) $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

2) $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$

3) If $f(x,y) \le g(x,y)\ \forall\ (x,y) \in D$,

$$\iint_D f(x,y)\, dA \le \iint_D g(x,y)\, dA$$

4) If $D = D' \cup D''$, where D' and D'' don't overlap except perhaps on their boundaries



$$\iint_D f\, dA = \iint_{D'} f\, dA + \iint_{D''} f\, dA$$

5) $\iint_D 1\, dA = A(D)$

6) If $m \le f(x,y) \le M\ \forall\ (x,y) \in D$, then

$$m A(D) \le \iint_D f(x,y)\, dA \le M A(D)$$

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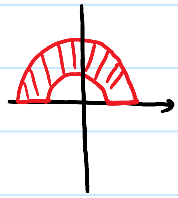
15.4 Double Integral in Polar coordinates

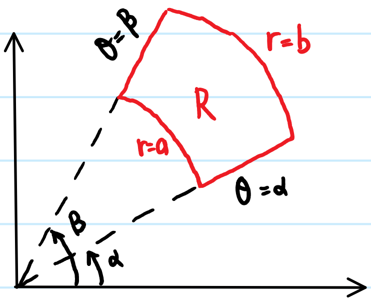
We may want to evaluate $\iint_D f(x,y)\,dA$, where D is



In this case the description of D in terms of rectangular coordinates can be quite complicated, but can be easily described using polar coordinates.

Then $D = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

This is special case of a polar rectangle $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$



Change to polar coordinates in Double Integral

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \in [0, 2\pi]$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example $\iint_R \frac{y^2}{x^2+y^2}\,dA$ where R is the region that lies between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $0<a<b$

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$$\iint \frac{y^2}{x^2+y^2}\, dA = \int_0^\pi \int_a^b r^2 \sin^2\theta \; r\, dr\, d\theta = \int_0^\pi \sin^2\theta \left[\frac{b^4}{4} - \frac{a^4}{4}\right] d\theta$$

$$= \left[\frac{b^4}{4} - \frac{a^4}{4}\right] \int_0^\pi \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \left[\frac{b^4 - a^4}{4}\right]\left[\frac{\pi}{2}\right] = \frac{\pi(b^4-a^4)}{8} \quad \blacksquare$$

Probability

A probability density function (pdf) f of a continuous random variable X is a function f s.t

1) $f(x) \geqslant 0$
2) $\int_{-\infty}^{\infty} f(x) = 1$
3) The probability that X lies betn a & b is found by integrating f from a to b i.e.

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

• Next, consider a pair of continuous random variables X & Y.

The joint density function of X & Y is a function f of two variable such that

1) $f(x,y) \geqslant 0$
2) $\iint_{\mathbb{R}^2} f(x,y)\,dA = 0$
3) The probability that (X,Y) lies in the region D is given by

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA.$$

Ex The joint density function for a pair of random variables X and Y is

$$f(x,y) = \begin{cases} Cx(1+y), & \text{if } 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{otherwise}. \end{cases}$$

a) Find C.

- To find C, we use the fact that the double integral of f is equal to 1.

As $f(x,y)=0$ outside the rectangle $[0,1] \times [0,2]$, we have

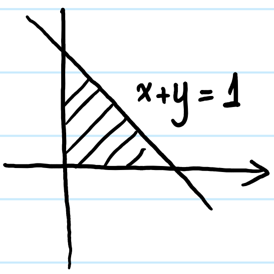
$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dA = \int_0^1\int_0^2 Cx(1+y)\,dy\,dx = C\int_0^1 x\left[y+\frac{y^2}{2}\right]_0^2 dx$$

$$= C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C$$

Therefore $2C = 1 \Rightarrow C = \frac{1}{2}$

b) Find $P(X+Y \le 1)$.

To compute this we need to determine the region D
s.t. $P(X+Y \le 1) = P((X,Y) \in D)$



Type 1 $\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

So, $P(X+Y \le 1) = \displaystyle\int_0^1\int_0^{1-x} \frac{1}{2}x(1+y)\,dy\,dx = \int_0^1 \frac{1}{2}x\left[y+\frac{y^2}{2}\right]_{y=0}^{y=1-x}$

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$$= \frac{1}{4}\int_0^1 x^3 - 4x^2 + 3x \, dx = \frac{1}{4}\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{5}{48}.$$

Suppose X is a random variable with pdf $f_1(x)$ and Y is a random variable with pdf $f_2(x)$. Then X and Y are called independent random variables if their joint density function is the product of their individual pdf i.e.

$$f(x,y) = f_1(x) \cdot f_2(y)$$

<u>Expected Value</u>

If X is a random variable with pdf f, then it's mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

If X and Y are random variables w/ joint density function f, we define the <u>X-mean</u> and <u>Y-mean</u> (expected values of X and Y)

$$\mu_X = \iint_{\mathbb{R}^2} x f(x,y)\, dA \quad \& \quad \mu_Y = \iint_{\mathbb{R}^2} y f(x,y)\, dA$$